Solution to Exercise 4.1.

We have

$$g = \frac{Gm}{r^2}, \quad r = \frac{1}{2} R_\odot, \quad m = \left(\frac{r}{R_\odot} \right)^3 M_\odot = \frac{1}{8} M_\odot,$$

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and hence

$$g = \frac{1}{2} \frac{GM_\odot}{R_\odot^2} = 137 \ \mathrm{N/kg}.$$

Solution to Exercise 4.2.

We have

$$m(r) = \frac{4}{3}\pi r^3 \rho,$$

$$\frac{dP}{dr} = -\rho g = -\frac{Gm\rho}{r^2} = -\frac{4}{3}\pi G \rho^2 r.$$

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Integrating in radial coordinates, we get

$$[P]_0^R = -\frac{4}{3}\pi G \rho^2 \frac{1}{2} R^2,$$

$$P_c = \frac{2}{3}\pi G \rho^2 R^2 = \frac{2}{3}\pi G \frac{M^2}{\left(\frac{4}{3}\pi R^3\right)^2} R^2 = \frac{3GM^2}{8\pi R^4}.$$

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From the equation of state of ideal gas,

$$T_c = \frac{\mu P_c}{\mathfrak{R}\rho} = \frac{\mu}{\mathfrak{R}} \frac{3GM^2}{8\pi R^4} \frac{\frac{4}{3}\pi R^3}{M} = \frac{GM}{2R}\frac{\mu}{\mathfrak{R}}.$$

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