Solution to Exercise 5.1.

Solving equation (5.10) for ρ_c , we get

$$\rho_c^{\frac{1-n}{n}} = \frac{4\pi G}{(n+1)K}\frac{R^2}{\xi_1^2}\,.$$

$$\rho_c^{\frac{1-n}{n}} = \frac{4\pi G}{(n+1)K}\frac{R^2}{\xi_1^2}\,\rho_c^{\frac{1-n}{n}} = \frac{4\pi G}{(n+1)K}\frac{R^2}{\xi_1^2}\,.$$

Doing the same with equation (5.13), we get

$$\rho_c^{\frac{3-n}{2n}} = (4\pi)^{1/2}\left[\frac{G}{(n+1)K}\right]^{3/2}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}^{-1} M\,.$$

$$\rho_c^{\frac{3-n}{2n}} = (4\pi)^{1/2}\left[\frac{G}{(n+1)K}\right]^{3/2}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}^{-1} M\,.$$

$$\rho_c^{\frac{3-n}{2n}} = (4\pi)^{1/2}\left[\frac{G}{(n+1)K}\right]^{3/2}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}^{-1} M\,.$$

Eliminating ρ_c between these two equations, we have

$$\left[\frac{4\pi G}{(n+1)K}\right]^{\frac{3-n}{2}} R^{3-n}\,\xi_1^{n-3}$$

$$= (4\pi)^{\frac{1-n}{2}}\left[\frac{G}{(n+1)K}\right]^{\frac{3-3n}{2}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}^{n-1} M^{1-n}\,.$$

$$\left[\frac{4\pi G}{(n+1)K}\right]^{\frac{3-n}{2}} R^{3-n}\,\xi_1^{n-3}$$

$$= (4\pi)^{\frac{1-n}{2}}\left[\frac{G}{(n+1)K}\right]^{\frac{3-3n}{2}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}^{n-1} M^{1-n}\,.$$

$$\left[\frac{4\pi G}{(n+1)K}\right]^{\frac{3-n}{2}} R^{3-n}\,\xi_1^{n-3}$$

$$= (4\pi)^{\frac{1-n}{2}}\left[\frac{G}{(n+1)K}\right]^{\frac{3-3n}{2}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}^{n-1} M^{1-n}\,.$$

This relation reduces to

$$4\pi\left[\frac{G}{(n+1)K}\right]^{n}\xi_1^{-n-1}\left(-\frac{d\theta}{d\xi}\right)_{\xi_1}^{1-n} M^{n-1}R^{3-n} = 1\,,$$

$$4\pi\left[\frac{G}{(n+1)K}\right]^{n}\xi_1^{-n-1}\left(-\frac{d\theta}{d\xi}\right)_{\xi_1}^{1-n}M^{n-1}R^{3-n}=1\,,$$

$$4\pi\left[\frac{G}{(n+1)K}\right]^{n}\xi_1^{-n-1}\left(-\frac{d\theta}{d\xi}\right)_{\xi_1}^{1-n}M^{n-1}R^{3-n}=1\,,$$

and we finally get

$$K=(4\pi)^{\frac{1}{n}}\frac{G}{n+1}\xi_1^{-\frac{n+1}{n}}\left(-\frac{d\theta}{d\xi}\right)_{\xi_1}^{\frac{1-n}{n}}M^{\frac{n-1}{n}}R^{\frac{3-n}{n}}\,.$$

Solution to Exercise 5.2.

Derivation of (5.15, 5.16) is obvious; equation (5.18) is derived from (5.17) by using K as given by equation (5.14) and ρ_c given by equation (5.16).

Solution to Exercise 5.3.

Substitute these solutions into the Lane-Emden equation (5.8). For $n=0$ we have

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\xi^2\frac{\xi}{3}\right)=-1=-\left(1-\frac{\xi^2}{6}\right)^0\,.$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\xi^2\frac{\xi}{3}\right)=-1=-\left(1-\frac{\xi^2}{6}\right)^0\,.\quad\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\xi^2\frac{\xi}{3}\right)=-1=-\left(1-\frac{\xi^2}{6}\right)^0\,.$$

For $n=1$ we have

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\sin\xi+\xi\cos\xi\right)$$

$$=-\frac{\cos\xi}{\xi^2}+\frac{\cos\xi}{\xi^2}-\frac{1}{\xi}\sin\xi=-\left(\frac{\sin\xi}{\xi}\right)^1.$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\sin\xi+\xi\cos\xi\right)$$

$$=-\frac{\cos\xi}{\xi^2}+\frac{\cos\xi}{\xi^2}-\frac{1}{\xi}\sin\xi=-\left(\frac{\sin\xi}{\xi}\right)^1.$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\sin\xi+\xi\cos\xi\right)$$

$$=-\frac{\cos\xi}{\xi^2}+\frac{\cos\xi}{\xi^2}-\frac{1}{\xi}\sin\xi=-\left(\frac{\sin\xi}{\xi}\right)^1.$$

For $n=5$ we have

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\frac{1}{2}\xi^2\frac{\frac{2}{3}\xi}{\left(1+\xi^2/3\right)^{3/2}}\right)$$

$$= -\frac{1}{\left(1+\xi^2/3\right)^{3/2}} + \frac{1}{3}\xi\frac{3}{2}\frac{\frac{2}{3}\xi}{\left(1+\xi^2/3\right)^{5/2}}$$

$$= \frac{-\left(1+\xi^2/3\right)+\xi^2/3}{\left(1+\xi^2/3\right)^{5/2}} = -\frac{1}{\left(1+\xi^2/3\right)^{5/2}} = -\theta^5.$$

Solution to Exercise 5.4.

We have

$$\rho_c = a_3 \frac{3M_\odot}{4\pi R_\odot^3} \simeq 8 \times 10^4 \, kg/m^3,$$

$$P_c = c_3 \frac{GM_\odot^2}{R_\odot^4} \simeq 1.2 \times 10^{16} \, N/m^2,$$

$$T_c = b_3 \frac{GM_\odot \mu m_H}{kR_\odot} \simeq 1.1 \times 10^7 \, K,$$

$$\rho_c = a_3 \frac{3M_\odot}{4\pi R_\odot^3} \simeq 8 \times 10^4 \, kg/m^3, \rho_c = a_3 \frac{3M_\odot}{4\pi R_\odot^3} \simeq 8 \times 10^4 \, kg/m^3,$$

$$P_c = c_3 \frac{GM_\odot^2}{R_\odot^4} \simeq 1.2 \times 10^{16} \, N/m^2, P_c = c_3 \frac{GM_\odot^2}{R_\odot^4} \simeq 1.2 \times 10^{16} \, N/m^2,$$

$$T_c = b_3 \frac{GM_\odot \mu m_H}{kR_\odot} \simeq 1.1 \times 10^7 \, K, T_c = b_3 \frac{GM_\odot \mu m_H}{kR_\odot} \simeq 1.1 \times 10^7 \, K,$$

using mean molecular weight given by equation (3.20) as

$$\mu \approx \frac{4}{3 + 5X - Z} \simeq 0.6 \, .$$

$$\mu \approx \frac{4}{3 + 5X - Z} \simeq 0.6 \, \mu \approx \frac{4}{3 + 5X - Z} \simeq 0.6 \, .$$

At the point where $r = R/2$,

$$\frac{\rho}{\rho_c} = \left[\theta\left(\xi_1/2\right)\right]^3 \simeq \theta^3\left(3.5\right) \simeq 0.28^3 \simeq 0.025\,,$$

$$\frac{P}{P_c} = \left[\theta\left(\xi_1/2\right)\right]^4 \simeq 0.28^4 \simeq 0.008\,,$$

$$\frac{T}{T_c} = \left[\theta\left(\xi_1/2\right)\right]^1 \simeq 0.28\,.$$